# CONSTRAINT PROGRAMMING

# Introduction

- Disadvantages of SAT solvers:
  - The range of problems that can be solved is limited
    - integer variables can not be represented easily and efficiently
    - not every constraint can easily and efficiently be rewritten in CNF:
      - numerical constraints $x_1 + x_2 + \cdots + x_n \geq 4$
      - graph constraints (“from node *x* node *y* can be reached”, “the shortest path from node *x* to node *y* may not be longer than *a*”)
    - dealing with optimization problems is not straightforward
  - The specification language is not very simple to use

# Constraint Programming

- **Constraint programming:** a programming paradigm in which a problem is specified declaratively in terms of high-level constraints, and solvers find solutions

“Constraint programming =

| | |
|---|---|
| Model | (by user) |
| + | |
| Search | (by solver)” |

# Non-boolean Variables & High-level Constraints

- variables

  $E_{11} \ldots E_{99}$

- variables have domains

  $E_{xy} = \{1 \ldots 9\}$

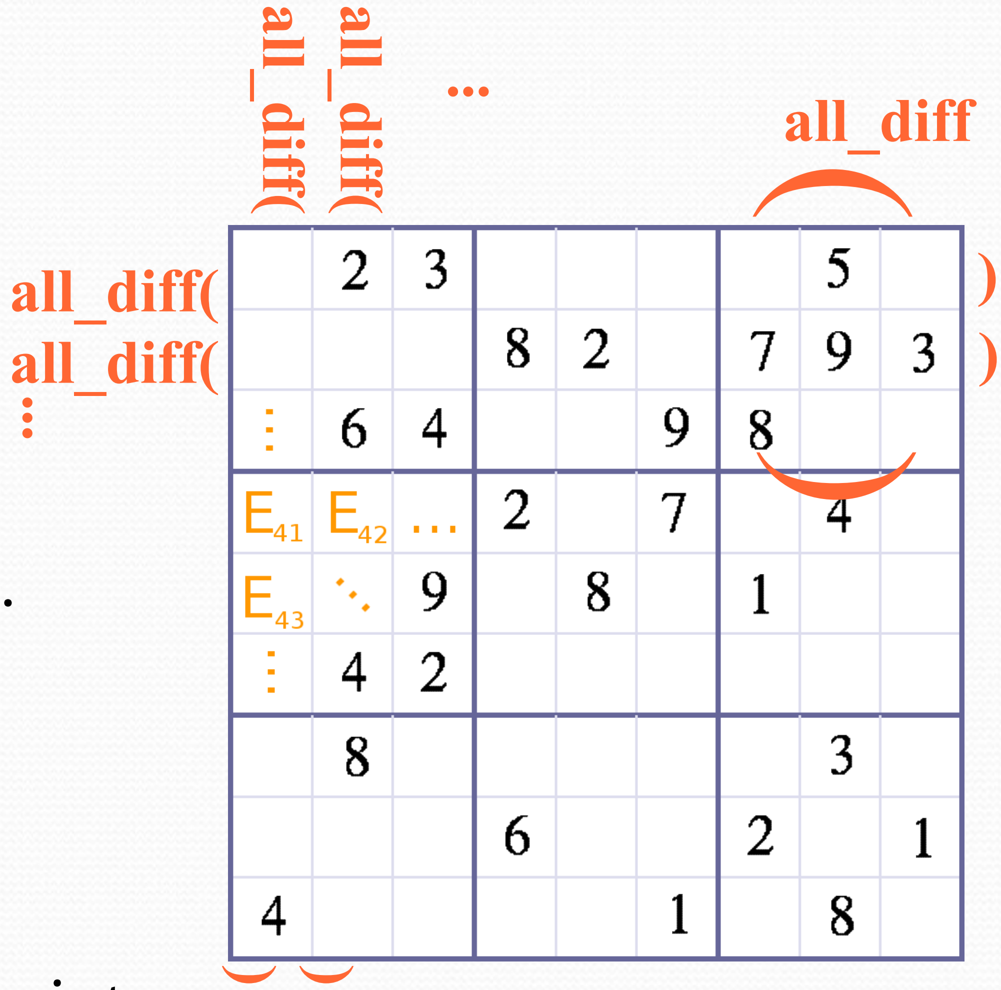
- Constraints

  all_different([$E_{1x}$]), …

  all_different([$E_{x1}$]),

  all_different([$E_{11} \ldots E_{33}$]), …

High-level all difference constraint



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | | | | | 5 | |
| | | | 8 | 2 | | 7 | 9 | 3 |
| ⋮ | 6 | 4 | | | 9 | 8 | | |
| $E_{41}$ | $E_{42}$ | … | 2 | | 7 | | 4 | |
| $E_{43}$ | ⋱ | 9 | | 8 | | 1 | | |
| ⋮ | 4 | 2 | | | | | | |
| | 8 | | | | | | 3 | |
| | | | 6 | | | 2 | | 1 |
| 4 | | | | | 1 | | 8 | |

# Solving

- Two approaches:
  - automatically translate high-level constraints into a low-level representation (like a CNF formula)
    - MiniZinc (specialized language) + G12 (solvers)
    - NumberJack (Python library)
  - run a solver which directly supports high-level constraints

**Domains must be finite**

Common in constraint programming are finite domain solvers based on exhaustive search & propagation

# Propagation

- Each (high-level) constraint is implemented in a **propagator**, which **only** operates on the variables listed in the constraint

- For each variable we store the **domain** of values the variable can still take, which may be
  - the complete domain (i.e., all values – clearly only works for problems with finite domains)

$$D(x) = \{ 2 \}, D(y) = \{ 2, 3 \}$$

  - lower and upper bounds, i.e. the minimum and maximal value the variable can still take

# Propagation

- The task of the propagator is to maintain **domain consistency**, i.e. to **shrink** the domains of variables to values that they can still take

if domain $D(x) = \{ 2 \}$, $D(y) = \{ 2, 3 \}$ and constraint $x \neq y$ apply, then we can deduce that $D(y) = \{3\}$.

Bounds

if domain $D(x) = \{ 1, \ldots, 5 \}$, $D(y) = \{ 1, 2 \}$ and constraint $x + y < 5$ apply then we deduce that $D(x) = \{ 1,\ldots, 3 \}$

# CP Search

**no domain to change any more**

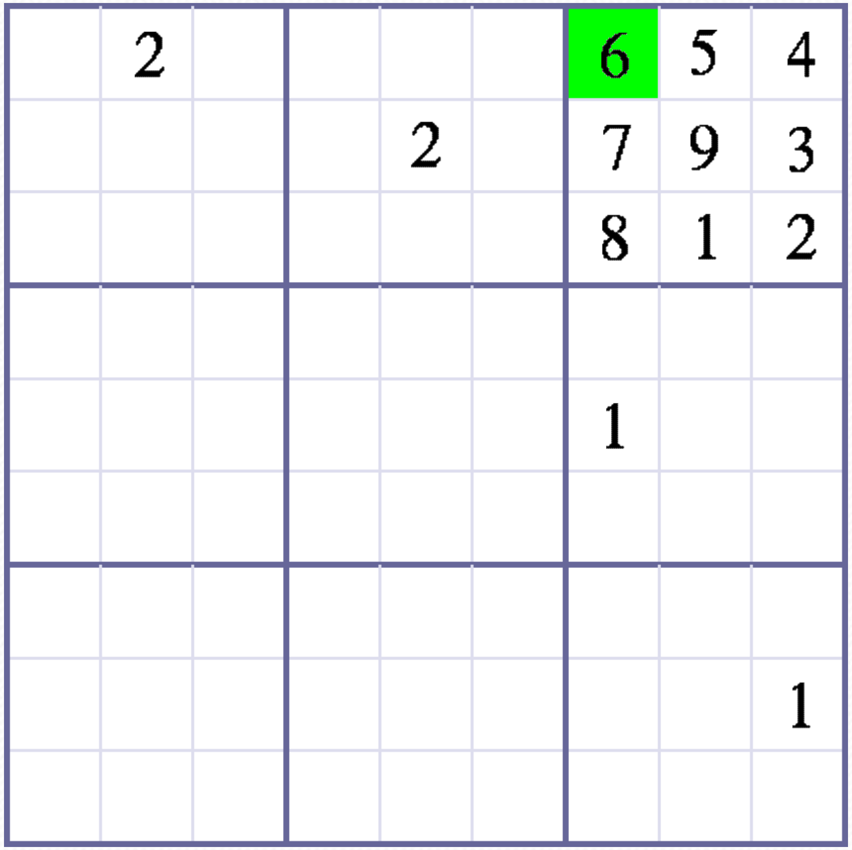
```
Search ( Variables ):
    propagate all constraints till fix point
    if contradiction found then return
    if at least one variable is not fixed yet then
        pick one variable V not fixed
        for each possible value of V do
            let V=value in this iteration
            Search ( Variables )
        od
    else
        print solution in Variables
```

# CP Search

all rows: all_different(row)
all columns: all_different(col)
all squares: all_different(square)

CP: Branch & Propagate

- propagate 2 (row)
- branch 4
- propagate 6 (square)



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 2 | | | | | 6 | 5 | 4 |
| | | | | 2 | | 7 | 9 | 3 |
| | | | | | | 8 | 1 | 2 |
| | | | | | | | | |
| | | | | | | 1 | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | 1 |
| | | | | | | | | |

# Propagation

- Propagators may implement special algorithms and data structures

<u>all-different constraint:</u>

all variables in a list must have a different value

<u>algorithm 1:</u> use inequality constraints independently

$D(x_1) = \{ 1, 2 \}$

$D(x_2) = \{ 1, 3 \}$

$D(x_3) = \{ 1, 3 \}$

$x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$

Propagation for inequality:
if one variable is fixed,
remove the corresponding
value from the domain
of the other variable
→ nothing happens in example

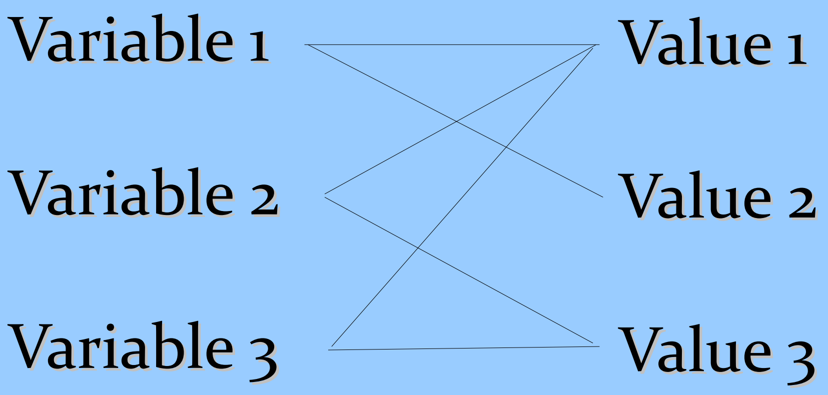
# Propagation

- Propagators may implement special algorithms and data structures

<u>all-different constraint:</u>

all variables in a list must have a different value

<u>algorithm 2:</u> graph-based; bipartite matching



Variable 1 is fixed to value 2

# Comparison to SAT solvers

- CP solvers support larger numbers of constraints & optimization
- When applied to CNF formulas, they search less efficiently as:
  - there is no clause learning
  - there is no propagation for pure symbols

These weaknesses led to the development of SMT SAT solvers (SAT-Modulo-Theories), which combine ideas of constraint programming and SAT solvers

Robert Nieuwenhuis, 2006.

# Implementation issues

- When to run a propagator?
  - when a variable changes? (In any way)
  - when one particular bound changes?

for domain $D(x) = \{ 1, 2, 3 \}$, $D(y) = \{ 1, 2, 3 \}$ and constraint $x + y < 5$; should we propagate when we remove value 1 from $D(y)$? When we remove value 3?

**In the CP literature, many different such strategies have been explored, called AC1, AC2, AC3, … AC5**

# Implementation issues

- Should we store simplified constraints during the search?

$D(x)=\{1,2,3\}$, $D(y) = \{ 4 \}$, $D(z) = \{ 1, 2\}$,
$x + y + z < 10 \rightarrow x + z < 6$

- Which order to select variables?
- Which order to select values?

# Implementation issues

- How to branch over variables?

$D(x)=\{1,\ldots,10\}$, $D(z) = \{ 1,\ldots,10\}$, $x + y < 20$

Branch with $D(x)=\{c\}$ *for all c in* $1\ldots10$*?*

Branch with $D(x)=\{1\ldots,5\}$ and $D(x)=\{6,..10\}$?

# INTEGER LINEAR PROGRAMMING

# Linear programming

- One special type of constraint is the linear constraint:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b$$

**Constant**

**Real valued variable**

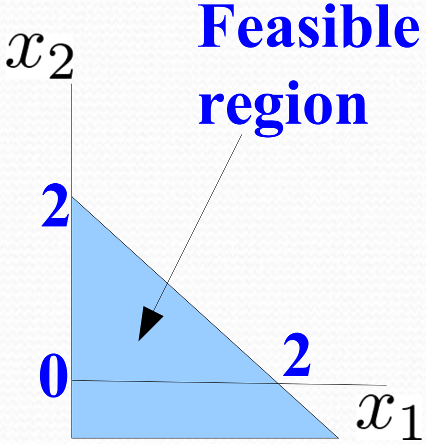
- A *linear program* is a constraint optimization problem on real-valued variables with a linear optimization criterion and linear constraints, and **no** other constraints:

$$\begin{array}{ll} \textbf{maximize} & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \textbf{where} & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\ & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \end{array}$$

# Linear Programming Examples

**maximize** $x_1 + 2x_2$
**where** $x_1 + x_2 \leq 2$
$-x_1 \leq 0$

$x_1 = 0, x_2 = 2$



**maximize** $-2x_1 + 3x_2$
**where** $x_1 - x_2 \leq 2$
$-x_1 \leq 0$
$x_2 \leq 3.5$

$x_1 = 0, x_2 = 3.5$

# Integer Linear Programming

- **Integer** linear programming differs from linear programming in that we constrain some variables to integer values; if some variables are not integer, this is also referred to as **mixed** integer linear programming

**maximize** $-2x_1 + 3x_2$

**where** $x_1 - x_2 \leq 2$

$-x_1 \leq 0$

$x_2 \leq 3.5$

$x_1, x_2 \in \mathbb{Z}$

$x_1 = 0, x_2 = 3$

# Solvers Linear Programming

- **Linear programs** can be solved in polynomial time by using *interior point* algorithms, which “walk through the interior of the feasible region”

- In practice, **linear programs** are often solved using *simplex* algorithms, which “walk over the outer rim of the feasible region” (the edges of the convex polytope)

# Solvers for
# Integer Linear Programming

- There are no polynomial solvers for **integer linear programming**
- Most solvers are based on *cut-and-branch*
  - solve the program without integer constraints
  - if solution is not integer, try to add a “clever” linear constraint that “cuts” the non-integer solution from the feasible solution space, without changing the feasible integer solutions
  - branch if no such linear constraint can be found, for the two closest integer values for one of the variables that does not have an integer value

# Graph Coloring using ILP

Example: we could use ILP to solve graph coloring with *k* colors

**(left: constraint in SAT form)** **(right: constaint in ILP form)**

- for each node *i*, create a formula

$$\phi_i = p_{i1} \vee p_{i2} \vee \cdots \vee p_{ik} \qquad x_{i1} + x_{i2} + \cdots + x_{ik} \geq 1$$

indicating that each node *i* must have a color

- for each node *i* and different pair of colors $c_1$ and $c_2$, create a formula

$$\phi_{ic_1c_2} = \neg p_{ic_1} \vee \neg p_{ic_2} \qquad (1 - x_{ic_1}) + (1 - x_{ic_2}) \geq 1$$

indicating a node may not have more than 1 color

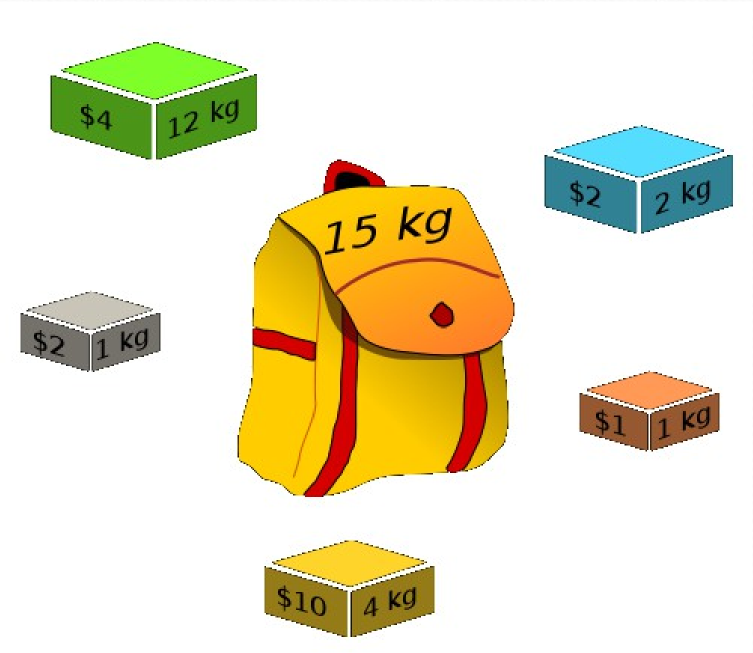
- for each edge, create *k* formulas

$$\phi_{ijc} = \neg p_{ic} \vee \neg p_{jc} \qquad (1 - x_{ic}) + (1 - x_{jc}) \geq 1$$

indicating that a pair of connected nodes *i* and *j* may not both have color *c* at the same time

- for each variable the requirement that its value can only be zero or one

# Knapsack using ILP



- **Given:**
  - $N$ items with sizes $a_1,\ldots,a_N$, prices $p_1,\ldots,p_N$
  - A maximum weight $W$
- **Find:**
  - a subset of items $I$ → *variables* $x_i$, *each with domain {0,1}*
- **Such that:**
  - $\sum_{i=1}^{n} p_i x_i$ is **maximal** (very valuable knapsack)
  - $\sum_{i=1}^{n} a_i x_i \leq W$ (knapsack with low weight)

**Also try: Set Cover**

# Comparison

| | Variables | Constraints | Optimization | Special Technology |
|---|---|---|---|---|
| SAT Solver | Boolean (0/1) | Clauses | Not supported directly | Clause learning, unit propagation, pure literals |
| CP Finite Domain Solver | Finite domain | Many | Many | Propagation |
| LP Solver | Real | Linear | Linear | Interior points, simplex |
| ILP Solver | Integer | Linear | Linear | Interior points, simplex, cut-and-branch |

# Comparison

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